ON A STRONG MULTIPLICITY ONE PROPERTY FOR THE LENGTH SPECTRA OF EVEN DIMENSIONAL COMPACT HYPERBOLIC SPACES

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Abstract. We prove a strong multiplicity one theorem for the length spectrum of compact even dimensional hyperbolic spaces i.e. if all but finitely many closed geodesics for two compact even dimensional hyperbolic spaces have the same length, then all closed geodesics have the same length.

1. Introduction

The analogy between the spectrum and arithmetic of hyperbolic surfaces was studied by A. Selberg. The primitive closed geodesics on a hyperbolic surface with finite volume can be considered as analogues of prime numbers and Selberg established an analogue of the prime number theorem for primitive closed geodesics (see [He]). The concept of the length spectrum can be introduced, and the Selberg trace formula establishes a relationship between the spectrum of the Laplacian acting on functions on a compact hyperbolic surface and the length spectrum of the surface. This can be generalized to higher dimensional compact hyperbolic spaces X_Γ which are quotients of the hyperbolic n-space $\mathbb{H}_n$ by torsion free uniform lattices Γ acting on $\mathbb{H}_n$ by isometries.

Define the length spectrum of X_Γ to be the function

$$L_\Gamma : \mathbb{R} \to \mathbb{N} \cup \{0\}$$

which to a real number l, assigns the number of closed geodesics of length l in X_Γ. Two compact hyperbolic spaces X_{Γ_1} and X_{Γ_2} are said to be *length-isospectral* if $L_{\Gamma_1}(l) = L_{\Gamma_2}(l)$ for all real numbers l.

In this article we establish the following strong multiplicity one type property for the length spectrum of even dimensional compact hyperbolic spaces:

Theorem 1. *Let Γ_1 and Γ_2 be uniform lattices in the isometry group of $2n$-dimensional hyperbolic space. Suppose $L_{\Gamma_1}(l) = L_{\Gamma_2}(l)$ for all but finitely many real numbers l.*

Then $L_{\Gamma_1}(l) = L_{\Gamma_2}(l)$ *for all real numbers* l *i.e., the corresponding compact hyperbolic spaces are length-isospectral.*

Remark 1. This result can be considered in analogy with the classical strong multiplicity one theorem for cusp forms. Suppose f and g are newforms for some Hecke congruence subgroup $\Gamma_0(N)$ such that the eigenvalues of the Hecke operator at a prime p are equal for all but finitely many primes p. Then the strong multiplicity one theorem of Atkin and Lehner states that f and g are equal (*cf.* [La, p.125]).

The proof of Theorem 1 uses the analytic properties of Ruelle zeta function, in particular the functional equation satisfied by it. The method is similar to the proof of the strong multiplicity one theorem for L-functions of the Selberg class given in [MM]. For odd dimensions, the form of the functional equation satisfied by the Ruelle zeta function is different, and hence our method does not shed any light when the dimension is odd.

Remark 2. Using an analytic version of the Selberg Trace formula, J. Elstrodt, F. Grunewald, and J. Mennicke proved a different version of Theorem 1 for $n = 2, 3$. ([EGM, Theorem 3.3, p.203]). Their notion of length spectrum is different from the notion defined in this paper.

2. Preliminaries

Let G be the connected component of the isometry group $\mathrm{SO}(n, 1)$ of $\mathbb{H}_n$. Fix a maximal compact subgroup K of G. Hence G/K is homeomorphic to the n-dimensional hyperbolic space $\mathbb{H}_n$. Let X_Γ be a compact n-dimensional hyperbolic space of the form

$$X_\Gamma = \Gamma\backslash G/K, \tag{1}$$

where Γ is a torsion free uniform lattice in G.

Let C be a free homotopy class in X_Γ. Let γ_C be a curve in C defined on the interval $[0, a]$, for some positive real number a. There exists a lift $\tilde{\gamma}_C$ of γ_C to $\mathbb{H}_n$. Since Γ acts on $\mathbb{H}_n$ by deck transformations, the action is transitive on fibres and hence there exists $g_C \in \Gamma$ such that $\tilde{\gamma}_C(a) = g_C.\tilde{\gamma}(0)$. The element g_C is determined upto conjugacy in Γ. For an element g of Γ, let $[g]$ denote the conjugacy class of g in Γ. It can be seen that the map $C \mapsto [g_C]$ is a bijection between the collection of free homotopy classes in X_Γ and the set of conjugacy classes of elements in Γ.

It is known that in any negatively curved compact Riemannian manifold every free homotopy class C contains a unique closed geodesic which we continue to denote by γ_C. Since X_Γ has constant negative sectional curvature -1, we have the following lemma :

Lemma 1. *There is a bijective correspondence between the set of closed geodesic classes in X_Γ and the set of conjugacy classes of Γ given by,*

$$[\gamma_C] \to [g_C] .$$

Definition 2.1. Let $\gamma \in \Gamma$. The length $\ell(\gamma)$ of the conjugacy class $[\gamma]$ is defined to be the length of the unique closed geodesic in the free homotopy class corresponding to γ in X_Γ.

Remark 3. Let $G = KAN$ be an Iwasawa decomposition for $G = SO(n, 1)$, where A is the connected component of the split part of a Cartan subgroup of G. Let $\mathfrak{a}$ be it's Lie algebra Fix an element X_0 of norm 1 in $\mathfrak{a}$ with respect to the Killing form. Let M denote the centralizer of A in K. It is known that every semisimple element γ of G is conjugate to an element $m_\gamma a_\gamma$ in G where $m_\gamma \in M$ and $a_\gamma \in A$. If γ is in Γ and the length of $[\gamma]$ is $\ell(\gamma)$, then $a_\gamma = \exp(\ell(\gamma)X_0$.

Since Γ is an uniform lattice in G, it is known that any relatively compact subset of G intersects only finitely many G-conjugacy classes of elements in Γ. Hence it follows that there are only finitely many closed geodesics of a fixed length.

We define the length spectrum of X_Γ to be the function L_Γ defined on $\mathbb{R}$ by,

$$L_\Gamma(l) = \text{The number of conjugacy classes } [\gamma] \text{ in } \Gamma \text{ such that } \ell(\gamma) = l.$$

3. Ruelle Zeta function

In this section, we recall the definition and some properties of the Ruelle Zeta function attached to a compact hyperbolic space.

Definition 3.1. A conjugacy class $[\gamma]$ of Γ is called primitive if $[\gamma] \neq [\delta^n]$ for any integer $n \geq 2$ and $\delta \in \Gamma$.

Define the primitive length spectrum of X_Γ to be the function $PL_\Gamma(l)$ defined by,

$$PL_\Gamma(l) = \text{The number of primitive conjugacy classes } [\gamma] \text{ in } \Gamma \text{ such that } \ell(\gamma) = l.$$

By P_Γ, we denote the set of primitive conjugacy classes in Γ. For $\gamma \in \Gamma$, define $N(\gamma) = e^{\ell\ (\gamma)}$. The Ruelle Zeta function is defined by the infinite product:

$$R_\Gamma(z) = \prod_{\gamma \in P_\Gamma} (1 - (N(\gamma))^{-z})^{(-1)^{(n-1)}} \tag{2}$$

It can be shown that the above product converges uniformly on compact sets in the right half plane $Re(z) > \rho$ (for some ρ which depends on n). Hence it defines a holomorphic function on the domain $Re(z) > \rho$. The following theorem describes the analytic properties of the Ruelle zeta function of relevance to us (see [BO, page 126, Theorems 4.3 and 4.4]):

Theorem 2. *The Ruelle Zeta function R_Γ admits a meromorphic continuation to the whole complex plane $\mathbb{C}$ and satisfies the following functional equation :*

- *X_Γ is even dimensional of dimension $2n$.*

$$R_\Gamma(z) R_\Gamma(-z) = \left[\sin\left(\frac{\pi z}{T}\right)\right]^{2n.\chi(X_\Gamma)} \tag{3}$$

where T is a positive constant independent of Γ and $\chi(X_\Gamma)$ is the Euler Characteristic of X_Γ.

- *X_Γ is odd dimensional of dimension $2n+1$.*

$$\frac{R_\Gamma(z)}{R_\Gamma(-z)} = \exp\left[\frac{(4\pi(n+1)\ vol(X_\Gamma))\, z}{TC}\right] \tag{4}$$

where $T,\ C$ are positive constants independent of Γ.

4. Proof of Theorem 1

We begin with the following preliminary lemma.

Lemma 2. *Let $e \neq \alpha \in \Gamma$. Then the centralizer $C_\Gamma(\alpha)$ is cyclic.*

Proof. Let h be an element of $C_\Gamma(\alpha)$. Then h is conjugate to an element $g = m_h a_h$ in G where $m_h \in M$ and $a_h \in A$. Let $H = \{a_h : h \in C_\Gamma(\alpha)\}$. Since elements of M commute with elements of A, it follows that H is a subgroup of A. Since Γ is discrete, so is H and consequently H is cyclic as A is isomorphic to $\mathbb{R}$. Let a_{h_0} be a generator of H. We show that $C_\Gamma(\alpha)$ is generated by h_0.

Given $h \in C_\Gamma(\alpha)$, $a_h = a_{h_0}^k$ for some $k \in \mathbb{Z}$. Then $h.h_0^{-k}$ is conjugate to some element of M. Since Γ is discrete and torsion-free, and M is compact, it follows that $h.h_0^{-k} = e$. Thus $C_\Gamma(\alpha)$ is cyclic. □

Corollary 1. *Let $[\alpha]$ and $[\beta]$ be primitive conjugacy classes such that $\left[\alpha^k\right] = \left[\beta^{k'}\right]$ for some natural numbers $k,\ k'$. Then $[\alpha] = [\beta]$ and $k = k'$.*

Proof. Let $[g] = \left[\alpha^k\right] = \left[\beta^{k'}\right]$. By Lemma 2 it follows that $C_\Gamma(g)$ is cyclic. Since both α and β are primitive conjugacy classes it follows that $[\alpha] = [\beta]$ and $k = k'$. □

Now we prove the following lemma, recovering the length spectrum from the primitive length spectrum :

Lemma 3.

$$L_\Gamma(l) = \sum_{d\,=\,1}^{\infty} PL_\Gamma\left(\frac{l}{d}\right) \quad \forall\ l \in \mathbb{R}.$$

Proof. Let $d \geq 0$ be an integer and l be a real number. Note that if $[\gamma]$ is a primitive conjugacy class of length l/d, then $\left[\gamma^d\right]$ is a conjugacy class of length l. Hence by Corollary 1,

$$L_\Gamma(l) \geq \sum_{d\,=\,1}^{\infty} PL_\Gamma\left(\frac{l}{d}\right) \quad \forall\ l \in \mathbb{R}.$$

Conversely, given a conjugacy class $[\gamma]$ of length l, the associated primitive class is of length l/d for some non-negative integer d. Hence the other inequality follows and this proves the lemma. □

Corollary 2. *Let Γ_1 and Γ_2 be torsion free uniform lattices in $SO(n,1)$. Let X_{Γ_1} and X_{Γ_2} be the associated compact hyperbolic spaces. Suppose*

$$PL_{\Gamma_1}(l) = PL_{\Gamma_2}(l) \quad \forall\ l \in \mathbb{R}.$$

Then

$$L_{\Gamma_1}(l) = L_{\Gamma_2}(l) \quad \forall\ l \in \mathbb{R}.$$

i.e. the spaces X_{Γ_1} and X_{Γ_2} are length isospectral.

In order to prove Theorem 1, by Corollary 2, it is enough to show that the primitive length spectra of X_{Γ_1} and X_{Γ_2} are equal. Let $R_{\Gamma_1}(z)$ and $R_{\Gamma_2}(z)$ be the respective Ruelle Zeta functions. Let dim X_{Γ_1} = dim X_{Γ_2} = $2n$. From the definition of the Ruelle Zeta function, we have in the region $\mathrm{Re}(z) > \rho$,

$$\frac{R_{\Gamma_1}(z)}{R_{\Gamma_2}(z)} = \frac{\prod_{\gamma\in P_{\Gamma_1}}\left(1-N(\gamma)^{-z}\right)^{(-1)^{(2n-1)}}}{\prod_{\gamma'\in P_{\Gamma_2}}\left(1-N(\gamma')^{-z}\right)^{(-1)^{(2n-1)}}} \tag{5}$$

Under the hypothesis of Theorem (1), all but finitely many factors cancel out. Hence there exist finite sets S_1 and S_2 such that for $\mathrm{Re}(z) > \rho$,

$$\frac{R_{\Gamma_1}(z)}{R_{\Gamma_2}(z)} = \frac{\prod_{j\in S_2}\left(1-N(\gamma_j')^{-z}\right)}{\prod_{i\in S_1}\left(1-N(\gamma_i)^{-z}\right)}.$$

Since the products in the above equation are over finite sets, the ratio $R_{\Gamma_1}(z)/R_{\Gamma_2}(z)$ defines a meromorphic function on the whole complex plane. Since $R_{\Gamma_1}/R_{\Gamma_2}$ admits a meromorphic continuation, the two expressions must agree for all $z \in \mathbb{C}$, i.e.

$$\frac{R_{\Gamma_1}(-z)}{R_{\Gamma_2}(-z)} = \frac{\prod_{j\in S_2}\left(1-N(\gamma_j')^{z}\right)}{\prod_{i\in S_1}\left(1-N(\gamma_i)^{z}\right)}$$

From the functional equation (3) applied to X_{Γ_1} and X_{Γ_2},

$$\begin{aligned}\frac{R_{\Gamma_1}(z)R_{\Gamma_1}(-z)}{R_{\Gamma_2}(z)R_{\Gamma_2}(-z)} &= \sin\left(\pi z/T\right)^{2n.(\chi(X_{\Gamma_1})\ -\ \chi(X_{\Gamma_2}))} \\ &= \frac{\prod_{j\in S_2}\left(1-N(\gamma_j')^{-z}\right)\left(1-N(\gamma_j')^{z}\right)}{\prod_{i\in S_1}\left(1-N(\gamma_i)^{-z}\right)\left(1-N(\gamma_i)^{z}\right)}\end{aligned} \tag{6}$$

If $\chi(X_{\Gamma_1}) - \chi(X_{\Gamma_2}) \neq 0$, then every integer multiple of T is either a zero or a pole of the left hand side in equation (6). On the other hand, all zeros and poles of right hand side are on imaginary axis. This leads to a contradiction. Hence we conclude that $\chi(X_{\Gamma_1}) = \chi(X_{\Gamma_2})$ and the left hand side in (6) is identically 1 i.e. for all $z \in \mathbb{C}$,

$$\prod_{i\in S_1}(1-N(\gamma_i)^{-z})\prod_{i\in S_1}(1-N(\gamma_i)^{z}) \ = \prod_{j\in S_2}(1-N(\gamma_j')^{-z})\prod_{j\in S_2}(1-N(\gamma_j')^{z})$$

The function on the left side of the above equation vanishes at Z if and only if $z = 2\pi ik/\ell(\gamma_j)$ for some $k \in \mathbb{Z}$ and $j \in S_1$. Similarly the function on the right side of the above equation vanishes at exactly when $z = 2\pi ik/\ell(\gamma_j')$ for some $k \in \mathbb{Z}$ and $j \in S_2$. Hence we get an equality of sets with multiplicity :

$$\left\{\frac{2\pi ik}{\ell(\gamma_j)} : k \in \mathbb{Z};\ j \in S_1\right\} \quad = \quad \left\{\frac{2\pi ik}{\ell(\gamma_j')} : k \in \mathbb{Z};\ j \in S_2\right\}.$$

Hence we conclude that all factors in equation (5) must cancel out. Consequently, $PL_{\Gamma_1}(l) = PL_{\Gamma_2}(l)$ for all $l \in \mathbb{R}$, and Theorem 1 follows from Corollary 2.

Remark 4. In the odd dimensional case, arguing as above and using the functional equation (4), yields the following equation:

$$\exp\left[\frac{(4\pi(n+1)(\mathrm{vol}(X_1)-\mathrm{vol}(X_2))z)}{TC}\right] = \frac{\prod\limits_{i\in S_1}\left(1-N(\gamma_i)^{-z}\right)\prod\limits_{j\in S_2}\left(1-N(\gamma_j')^{-z}\right)}{\prod\limits_{i\in S_1}\left(1-N(\gamma_i)^{z}\right)\prod\limits_{j\in S_2}\left(1-N(\gamma_j')^{z}\right)}$$

Since

$$\frac{1-a^z}{1-a^{-z}} = -a^z,$$

the above equation yields no information. Hence we cannot conclude anything in the odd dimensional case.

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